

RESOLVABILITY OF PSEUDOCOMPACT SPACES

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ABSTRACT. Given cardinals λ and μ we say that

$[\mathbf{B}(\lambda)]^{\mathbf{C}}$ is μ -colorable

if there is a coloring $f : \mathbf{B}(\lambda) \rightarrow \mu$ such that $f''Z = \mu$ whenever a subspace $Z \subset \mathbf{B}(\lambda)$ is homeomorphic to the Cantor set, where $\mathbf{B}(\lambda)$ denotes the Baire space of weight λ .

We prove that a crowded feebly compact regular space X is μ -resolvable provided $[\mathbf{B}(\lambda)]^{\mathbf{C}}$ is μ -colorable for each $\lambda < \hat{c}(X)$.

Consequently,

- (a) every crowded pseudocompact space X with $c(X) < (2^\omega)^{+\omega}$ is 2^ω -resolvable;
- (b) if $V = L$, then every crowded pseudocompact space is 2^ω -resolvable.

A Tychonoff space X is said to be *pseudocompact* if every continuous real-valued function on X is bounded (see [2]). The notion of feebly compact spaces was introduced by Mardešić in [3]: a topological space is *feebly compact* if every locally finite family of open sets is finite. He proved that a Tychonoff space is feebly compact if and only if it is pseudocompact.

In [6] Pavlov quoted the following two questions:

- (a) Is there an infinite regular feebly compact irresolvable space?
- (b) Is there an infinite Tychonoff pseudocompact irresolvable space?

In [4] van Mill proved that every crowded pseudocompact space with countable cellularity is 2^ω -resolvable. In [5] Ortiz-Castillo and Tomita showed that a crowded pseudocompact space with cellularity at most 2^ω is resolvable.

To formulate our result we need the following notions and notations.

If X and Y are topological spaces, and μ is a cardinal, we let

$$[X]^Y = \{Z \subset X : Z \text{ is homeomorphic to } Y\}.$$

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We say that a family $\mathcal{Z} \subset \mathcal{P}(X)$ is μ -colorable if there is a function $f : X \rightarrow \mu$ such that $f''Z = \mu$ for each $Z \in \mathcal{Z}$.

Let $\mathbf{B}(\lambda)$ denote the Baire space of weight λ , and \mathbf{C} the Cantor space.

The family of the non-empty regular open subsets of a space X is denoted by $\text{RO}^+(X)$. A topological space is π -regular if every open set contains a non-empty regular closed set.

Based on ideas of van Mill [4], and of Ortiz-Castillo and Tomita [5] we prove the following theorem.

Theorem 1. *A crowded feebly compact π -regular T_2 space X is μ -resolvable provided $\text{Col}_\mu([\mathbf{B}(\lambda)]^{\mathbf{C}})$ holds for each cardinal $\lambda < \hat{c}(X)$.*

Proof. Write $\kappa = \hat{c}(X)$.

The boundary of a subspace $Z \subset X$ will be denoted by $\text{bd}(Z)$, $\text{bd}(Z) = \overline{Z} \setminus \text{int}(Z)$.

Definition 2. Let

$$\mathfrak{U} = \{ \langle U_n : n < \omega \rangle \in {}^\omega(\tau_X) : \overline{U_{n+1}} \subset U_n \text{ for each } n \in \omega \}.$$

For $\mathcal{U} = \langle U_n : n < \omega \rangle \in \mathfrak{U}$ let

$$\cap \mathcal{U} = \bigcap \{ U_n : n \in \omega \} \text{ and } \partial(\mathcal{U}) = \text{bd}(\cap \mathcal{U}).$$

Since $\bigcap \{ U_n : n \in \omega \} = \bigcap \{ \overline{U_n} : n \in \omega \}$ is closed we have

$$\partial(\mathcal{U}) = \cap \mathcal{U} \setminus \text{int}(\cap \mathcal{U}).$$

Lemma 3. *If $\mathcal{U} = \langle U_n : n < \omega \rangle \in \mathfrak{U}$ and $V \in \tau_X$ such that $(U_n \setminus \overline{U_{n+1}}) \cap V \neq \emptyset$ for each $n \in \omega$ then $\partial(\mathcal{U}) \cap \overline{V} \neq \emptyset$.*

Proof of the lemma. Let $W_n = (U_n \setminus (\cap \mathcal{U})) \cap V$. Then W_n is open, and $W_n \supset (U_n \setminus \overline{U_{n+1}}) \cap V \neq \emptyset$ by assumption.

Since X is feebly compact, the family $\{W_n : n \in \omega\}$ is not locally finite, i.e., there is $x \in X$ such that every neighborhood of x intersects infinitely many W_n . Hence $x \in \overline{W_n}$ for all $n \in \omega$ because the sequence W_0, W_1, \dots is decreasing.

Since $W_n \cap (\cap \mathcal{U}) \subset W_n \cap U_{n+1} = \emptyset$, we have $\overline{W_n} \cap \text{int}(\cap \mathcal{U}) = \emptyset$. Thus

$$\begin{aligned} x \in \bigcap_{n \in \omega} \overline{W_n} &\subset \bigcap_{n \in \omega} \left(\overline{(U_n \setminus \text{int}(\cap \mathcal{U})) \cap V} \right) \subset \\ &\left(\bigcap_{n \in \omega} \overline{U_n} \setminus \text{int}(\cap \mathcal{U}) \right) \cap \overline{V} = \left(\bigcap_{n \in \omega} U_n \setminus \text{int}(\cap \mathcal{U}) \right) \cap \overline{V} = \partial(\mathcal{U}) \cap \overline{V}, \end{aligned}$$

and so $\partial(\mathcal{U}) \cap \overline{V} \neq \emptyset$. □

Following [5] we will define a tree $\mathcal{T} = \langle T, \supset \rangle$, where $\mathcal{T} \subset \text{RO}^+(X)$. We will construct the levels T_α of \mathcal{T} for $\alpha < \kappa$ by transfinite recursion on α such that

- (1) $T_0 = \{X\}$;
- (2) $\overline{U} \cap \overline{V} = \emptyset$ for each $\{U, V\} \in [T_\alpha]^2$;
- (3) for each $U \in T_\alpha$ the family $\mathcal{V}_U = \{V \in T_{\alpha+1} : \overline{V} \subset U\}$ has at least two elements and $\overline{U} = \overline{\bigcup \mathcal{V}_U}$;
- (4) if α is a limit ordinal, then $V \in T_\alpha$ iff there is a sequence $\langle U_\beta : \beta < \alpha \rangle$ such that $U_\beta \in T_\beta$ and $V = \text{int} \bigcap \{U_\beta : \beta < \alpha\} \neq \emptyset$.

If α is a limit ordinal, and $\langle U_\beta : \beta < \alpha \rangle$ is a sequence such that $U_\beta \in T_\beta$ and $V = \text{int} \bigcap \{U_\beta : \beta < \alpha\}$, then $\bigcap \{U_\beta : \beta < \alpha\} = \bigcap \{\overline{U_\beta} : \beta < \alpha\}$, hence V is a regular open set. So $T_\alpha \subset \text{RO}^+(X)$, and so the transfinite construction of the tree \mathcal{T} can be carried out.

It might happen that $\text{height}(\mathcal{T}) < \kappa$. In that case $T_\alpha = \emptyset$ for each α with $\text{height}(\mathcal{T}) \leq \alpha < \kappa$.

Write

$$T_\alpha = \{U_\xi^\alpha : \xi < \lambda_\alpha\},$$

where $\lambda_\alpha < \kappa$.

The tree $\mathcal{T} = \langle T, \supset \rangle$ can not contain $\hat{c}(X)$ -branches because given any branch $\langle U_\alpha : \alpha < \delta \rangle$, the family

$$\{U_\alpha \setminus \overline{U_{\alpha+1}} : \alpha + 1 < \delta\}$$

is a family of pairwise disjoint non-empty open sets by (3).

For each $\alpha < \hat{c}(X)$ with $cf(\alpha) = \omega$ fix a strictly increasing sequence $\vec{\alpha} = \langle \alpha_n : n < \omega \rangle$ of ordinals converging to α , let

$$\kappa_\alpha = \sup_{n \in \omega} \lambda_{\alpha_n}.$$

Since $\hat{c}(X) > \omega$ is a regular cardinal, we have $\kappa_\alpha < \hat{c}(X)$.

Let

$$Y_\alpha = \bigcup \{\partial(\mathcal{U}) : \mathcal{U} = \langle U_{\zeta_n}^{\alpha_n} : n < \omega \rangle \in \mathfrak{U}\}.$$

Clearly $Y_\alpha \cap Y_\beta = \emptyset$ for $\alpha < \beta$ because $Y_\beta \subset \bigcup T_\alpha$.

Let $r_\alpha : {}^\omega \kappa_\alpha \rightarrow \mu$ witness that $[\mathbf{B}(\kappa_\alpha)]^\mathbf{C}$ is μ -colorable. Define $h_\alpha : Y_\alpha \rightarrow \mu$ as follows: if $\mathcal{U} = \langle U_{\alpha_n, \zeta_n} : n < \omega \rangle \in \mathfrak{U}$, where $\vec{\alpha} = \langle \alpha_n : n < \omega \rangle$, then let

$$h_\alpha(p) = r_\alpha(\langle \zeta_n : n < \omega \rangle) \text{ for each } p \in \partial(\mathcal{U}). \quad (\circ)$$

Define the partial function $h : X \rightarrow \mu$ as follows:

$$h = \bigcup_{\alpha < \lambda^+} h_\alpha.$$

We show that

$$h \text{ witnesses that } X \text{ is } \mu\text{-resolvable.} \quad (+)$$

Write $T_\kappa = \emptyset$ and for each non-empty open $U \subset X$ define the ordinal $\gamma_U \leq \kappa$ by the formula

$$\gamma_U = \min\{\alpha : U \cap \bigcup T_\alpha = \emptyset\}. \quad (\star)$$

The ordinal γ_U is a limit ordinal because of (3).

Since X is π -regular, every non-empty subset G of X contains a non-empty open W such that $\overline{W} \subset G$ and $\gamma_{W'} = \gamma_W$ for all non-empty open $W' \subset W$.

Thus the following lemma yields (+) and so it concludes the proof of the theorem.

Lemma 4. *If $W \in \tau_X^+$ such that $\gamma_{W'} = \gamma_W$ for all non-empty open $W' \subset W$ then there is $\alpha \leq \gamma_W$ such that $h_\alpha'' \overline{W} = \mu$.*

Proof of Lemma 4. Write $\gamma = \gamma_W$.

Since $\gamma_{W'} = \gamma$ for each non-empty open $W' \subset W$, for each $\alpha < \gamma$ we have

$$\bigcup T_\alpha \cap W \subset^{dense} W. \quad (\dagger)$$

Next we need to prove Claims 5–8 below.

Claim 5. *Assume that $\beta < \gamma$, $U \in T_\beta$, $\emptyset \neq V \subset U \cap W$ open. Then there is an ordinal $\alpha_{V,U}$ such that $\beta < \alpha_{V,U} < \gamma$ and for each α with $\alpha_{V,U} < \alpha < \gamma$ the family $\{U' \in T_\alpha : U' \cap V \neq \emptyset\}$ has at least two elements.*

Proof of Claim 5. First we show that

- (a) *there is an ordinal $\alpha_{V,U}$ such that $\beta < \alpha_{V,U} < \gamma$ and the family $\{U' \in T_{\alpha_{V,U}} : U' \cap V \neq \emptyset\}$ has at least two elements, U^0 and U^1 .*

If not, then by (\dagger) for each α with $\beta < \alpha < \gamma$ there is exactly one $U_\alpha \in T_\alpha$ such that $V \cap U_\alpha \neq \emptyset$.

Then $\langle U_\alpha : \beta < \alpha < \gamma \rangle$ is a chain in \mathcal{T} .

Moreover, by (\dagger) , we have $\overline{V} = \overline{U_\alpha \cap V}$ and so

$$\overline{V} \subset \bigcap_{\beta < \alpha < \gamma} U_\alpha.$$

Thus $\text{int} \bigcap_{\beta < \alpha < \gamma} U_\alpha \in T_\gamma$ and so $V \cap \bigcup T_\gamma \neq \emptyset$, which contradicts the choice of γ in (\star) . Thus (a) holds.

Now let $\alpha_{V,U} \leq \alpha < \gamma$ be arbitrary.

Then, by (\dagger) there are $U_\alpha^0, U_\alpha^1 \in T_\alpha$ with $(U^i \cap V) \cap U_\alpha^i \neq \emptyset$. Then $U^i \supset U_\alpha^i$ for $i = 0, 1$, and so $U_\alpha^0 \neq U_\alpha^1$ as required. \square

Claim 6. *There is a strictly increasing sequences $\langle \beta_m : m < \omega \rangle$ of ordinals converging some ordinal $\alpha \leq \gamma$, and there is an order preserving injection f from the Cantor tree $2^{<\omega}$ into \mathcal{T} such that*

$$f(s) \in T_{\beta_{|s|}} \text{ and } f(s) \cap W \neq \emptyset \text{ for each } s \in 2^{<\omega}. \quad (\bullet)$$

Proof of the Claim 6. We construct β_m and $f \upharpoonright {}^m 2$ by induction on m .

Let $\beta_0 = 0$ and $f(\emptyset) = X$.

Assume that we have β_{m-1} and $f(s) \in T_{\beta_{m-1}}$ for $s \in {}^{m-1}2$ with $f(s) \cap W \neq \emptyset$. Using the notation of Lemma 5 let

$$\beta_m = \max\{\alpha_{f(s) \cap W, f(s)} : s \in {}^{m-1}2\} < \gamma.$$

Then for each $s \in {}^{m-1}2$ there are two distinct elements $f(s \smallfrown 0)$ and $f(s \smallfrown 1)$ of T_{β_m} such that $f(s \smallfrown i) \cap (W \cap f(s)) \neq \emptyset$ for $i = 0, 1$. Then $f(s \smallfrown i) \cap f(s) \neq \emptyset$ implies $f(s \smallfrown i) \subset f(s)$, so f will be an order-preserving injection.

Since the inductive step can be carried out, we proved the lemma. \square

Definition 7. If $\alpha < \hat{c}(X)$ is an ordinal with $cf(\alpha) = \omega$, and $y \in {}^\omega \kappa_\alpha$ let

$$\partial^\alpha(y) = \partial(\langle U_{y(n)}^{\alpha_n} : n \in \omega \rangle)$$

provided $\langle U_{y(n)}^{\alpha_n} : n \in \omega \rangle \in \mathfrak{U}$, i.e. $\langle U_{y(n)}^{\alpha_n} : n \in \omega \rangle$ is a chain in \mathcal{T} ; let $\partial^\alpha(y) = \emptyset$ otherwise.

Claim 8. *There is an ordinal $\alpha \leq \gamma$ and there is a Cantor set $C \subset \mathbf{B}(\kappa_\alpha)$ such that*

$$\partial^\alpha(y) \cap \overline{W} \neq \emptyset$$

for each $y \in C$.

Proof of Claim 8. By Claim 6 there is a strictly increasing sequences $\langle \beta_m : m < \omega \rangle$ of ordinals converging some ordinal $\alpha \leq \gamma$, and there is an order preserving injection

$$f : \langle 2^{<\omega}, \subset \rangle \rightarrow \langle T, \supset \rangle$$

such that

$$f(s) \in T_{\beta_{|s|}} \text{ and } f(s) \cap W \neq \emptyset \text{ for each } s \in 2^{<\omega}.$$

For that ordinal α we fixed a sequence $\vec{\alpha} = \langle \alpha_n : n < \alpha \rangle$ converging to α .

We define an injective function F mapping the branches of the Cantor tree $\langle 2^{<\omega}, \subset \rangle$ into $\mathbf{B}(\kappa_\alpha)$ as follows.

If $x \in {}^\omega 2$ is a branch of the Cantor tree $2^{<\omega}$, then

$$\langle f(x \upharpoonright m) : m \in \omega \rangle$$

is an ordered set in the tree $\mathcal{T} = \langle T, \supset \rangle$, and it determines a branch

$$\langle U_{\zeta^x(\nu)}^\nu : \nu < \alpha \rangle$$

in the subtree $\mathcal{T}_{<\alpha} = \langle \bigcup_{\beta < \alpha} T_\beta, \supset \rangle$ such that

$$U_{\zeta^x(\beta_m)}^{\beta_m} = f(x \upharpoonright m)$$

for all $m \in \omega$.

If $n \in \omega$, then $\alpha_n < \alpha$, and so we can declare that

$$F(x)(n) = \zeta^x(\alpha_n).$$

Then

$$\{U_{\alpha_n, F(x)(n)} : n \in \omega\} \cup \{f(x \upharpoonright m) : m \in \omega\}$$

is a chain in \mathcal{T} . So

$$\partial(\langle f(x \upharpoonright m) : m \in \omega \rangle) = \partial(\langle U_{F(x)(n)}^{\alpha_n} : n \in \omega \rangle) = \partial^\alpha(F(x))$$

and $\partial(\langle f(x \upharpoonright m) : m \in \omega \rangle) \cap \overline{W} \neq \emptyset$ by Lemma 3. Thus

$$\partial^\alpha(F(x)) \cap \overline{W} \neq \emptyset \text{ for each } x \in 2^{<\omega}. \quad (\circ\circ)$$

To show that F is injective, pick $x \neq x' \in {}^\omega 2$.

Choose first m such that $x \upharpoonright m \neq x' \upharpoonright m$. Then $f(x \upharpoonright m) \neq f(x' \upharpoonright m)$ because f is injective. Then pick $n \in \omega$ such that $\alpha_n \geq \beta_m$.

Then $U_{F(x)(n)}^{\alpha_n} \subset f(x \upharpoonright m)$ and $U_{F(x')(n)}^{\alpha_n} \subset f(x' \upharpoonright m)$ so $F(x)(n) \neq F(x')(n)$ because $f(x \upharpoonright m) \cap f(x' \upharpoonright m) = \emptyset$.

Thus $F(x) \neq F(x')$, i.e. F is injective.

Consider the natural compact topology τ on ${}^\omega 2$.

The function F is continuous. Indeed, if $x, x' \in {}^\omega 2$, $x \upharpoonright m = x' \upharpoonright m$, and $\alpha_n \leq \beta_m$, then $F(x)(n) = F(x')(n)$.

So the function F is an isomorphism because the Cantor space $({}^\omega 2, \tau)$ is compact and the function F is injective and continuous from $({}^\omega 2, \tau)$ into the Hausdorff space $\mathbf{B}(\kappa_\alpha)$. So the set

$$C = \text{ran}(F)$$

is a Cantor subset of the metric space $\mathbf{B}(\kappa_\alpha)$ and C meets the requirements by $(\circ\circ)$. \square

Now we are ready to complete the proof of Lemma 4.

By Claim 8 there is an ordinal $\alpha \leq \gamma$ and there is a Cantor set C in $\mathbf{B}(\kappa_\alpha)$ such that

$$\partial^\alpha(y) \cap \overline{W} \neq \emptyset$$

for each $y \in C$.

Let $\zeta < \mu$. By the choice of the function r_α there is $y \in C$ such that $r_\alpha(y) = \zeta$. Then $\partial^\alpha(y) \cap \overline{W} \neq \emptyset$, and if $p \in \partial^\alpha(y) \cap \overline{W} \neq \emptyset$ then $h_\alpha(p) = r_\alpha(y) = \zeta$ by (o), so $\zeta \in h_\alpha'' \overline{W}$. \square

As we observed, Lemma 4 completes the proof of the theorem 1. \square

Corollary 9. (i) *Every crowded pseudocompact space X with $c(X) < (2^\omega)^{+\omega}$ is 2^ω -resolvable.*

(ii) *It is consistent that every crowded pseudocompact space is 2^ω -resolvable.*

Proof. Putting together [1, Theorem 3.5] and [1, Corollary 1.9] we obtain the following statement:

(HJS) *Let Y be a Hausdorff topological space with $|Y| = \nu$. Then $[Y]^C$ is 2^ω -colorable if either*
 (a) $\nu \leq (2^\omega)^{+\omega}$
 or
 (b) *for every cardinal μ with $2^\omega < \mu < \nu$ and $cf(\mu) = \omega$ we have $\mu^\omega = \mu^+$ and \square_μ .*

(i) Let X be a pseudoradial space X with $c(X) < (2^\omega)^{+\omega}$. If $\lambda < \hat{c}(X) < (2^\omega)^{+\omega}$ then $|B(\lambda)| = \lambda^\omega = \max(2^\omega, \lambda) < (2^\omega)^{+\omega}$, so $[B(\lambda)]^C$ is 2^ω -colorable by (HJS)(a). Thus we can apply Theorem 1 to show that X is 2^ω -resolvable.

(ii) Assume that for every cardinal μ with $2^\omega < \mu < \nu$ and $cf(\mu) = \omega$ we have $\mu^\omega = \mu^+$ and \square_μ (for example, $V = L$.) Then $[B(\lambda)]^C$ is 2^ω -colorable for each λ by (HJS)(b). Thus we can apply Theorem 1 to show that every pseudocompact space is 2^ω -resolvable. \square

REFERENCES

- [1] Hajnal, A.; Juhász, I.; Shelah, S. *Splitting strongly almost disjoint families*. Trans. Amer. Math. Soc. 295 (1986), no. 1, 369–387.
- [2] Hewitt, Edwin *Rings of real-valued continuous functions. I*. Trans. Amer. Math. Soc. 64, (1948). 45–99.
- [3] Mardešić, Sibe; Papić, Pavle, *Sur les espaces dont toute transformation réelle continue est bornée*. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 10 (1955), 225–232.
- [4] van Mill, Jan; *Every crowded pseudocompact ccc space is resolvable*. Topology Appl. 213 (2016), 127–134.
- [5] Yasser F. Ortiz-Castillo, Artur H. Tomita *Crowded pseudocompact Tychonoff spaces of cellularity at most the continuum are resolvable*, Toposym 2016, conference talk
http://www.toposym.cz/slides/slides-Ortiz_Castillo-2435.pdf.
- [6] Oleg Pavlov *Problems on (ir)resolvability*, in *Open problems in topology. II*. ed. Elliott Pearl. Elsevier B. V., Amsterdam, 2007

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